TWO-VARIATE RATIO TYPE ESTIMATORS IN DOUBLE SAMPLING FOR TWO STAGE DESIGNS

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(Received : September, 1974)

When information on some auxiliary character (x) is available or can be collected cheaply for all units of a population, various procedures, using such information are available for estimating the mean/total of the character under study (y). Ratio method of estimation is useful when the auxiliary character is highly positively correlated with the character under study and its mean is not equal to or near to zero. Two familiar ratio estimators are $\frac{\bar{y}_n}{\bar{x}_n}\bar{x}_N$ and $\bar{r}_n\,\bar{x}_n$, where $\bar{y}_n$ and $\bar{x}_n$ are the sample means of y and x respectively ; $\bar{r}_n=\frac{1}{n}\sum_{t}^{n}\frac{y_i}{x_i}$ and $\bar{x}_N$ is the population mean of x. Both these estimators are biased. Hartley and Ross [2] obtained almost unbiased ratio estimators of $\bar{y}_N$, the population mean of y, from a simple random sample (s.r.s.) drawn with replacement. Garg and Pillai [1] derived the variance of unbiased one auxiliary variate ratio estimator for two stage design using the technique of symmetric means.

In this paper variances of unbiased two-auxiliary variate (x_1 and x_2) ratio estimators of $\bar{y}_N$ have been derived in a two stage design using the technique of symmetric means developed by Tukey [5] and extended by Robson [4] and Garg and Pillai [1] where (i) population means of x_1 and x_2 are known, and (ii) the population means of x_1 and x_2 are not known (double sampling).

The technique of symmetric means and multiplication formula for the product of two symmetric means is as follows. The polynomial

$$\frac{1}{(n)_r} \sum_{(j_1 \neq j_2 \neq \ldots \neq)_r} \left(x_{1j_1}^{a_{11}} x_{2j_1}^{a_{21}} \ldots\ldots x_{mj_1}^{a_{m1}} \right) \ldots\ldots \left(x_{1j_r}^{a_{1r}} x_{2j_r}^{a_{2r}} \ldots x_{mj_r}^{a_{mr}} \right)$$

in the mn variates x_{ij} $i=1, 2, \ldots, m$; $j=1, 2\ldots\ldots, n$ is the multivariate symmetric mean and is devoted by

$$<(d_r)>=\left(a_1^* \right) \left(a_2^* \right) \cdots\cdots \left(a_r^* \right)$$

where $\left(a_1^* \right)$ is vector $(a_{1i}, a_{2i}, \ldots\ldots a_{mi})$

Let $$<(\alpha_r)>=<\left(a_1^* \right) \left(a_2^* \right) \cdots\cdots \left(a_r^* \right)>$$

and $$<(\beta_s)>=<\left(b_1^* \right) \left(b_2^* \right) \cdots\cdots \left(b_s^* \right)>$$

be the two symmetric means.

Let $$\rho_t(\alpha_r \beta_s)=[(a_{i1}+b_{j1})\ldots(a_{it}+b_{jt}), (a_{it+1}), \ldots (a_{ir}), (b_{jt+1})\ldots(b_{js})]$$

be obtained by pairing and adding t elements of α_r with t elements of β_s and let $R_t(\alpha_r, \beta_s)=\rho_t(\alpha_r, \beta_s)$ be the collection of all $t!\binom{r}{t}\times\binom{s}{t}$ possible sets $\rho_t(\alpha_r, \beta_s)$. Then the product of two symmetric means is given by

$$<(\alpha_r)><(\beta_s)>=\frac{1}{(n)_r\,(n)_s} \sum_{t=0}^{r} (n)_{r+s-t} \sum_{R_t} <\rho_t(\alpha_r \beta_s)>, \; r \leqslant s$$

It can be seen that if these mn variates represent a simple random sample of n observations from an m-dimensional finite population of size N, the expected value of $<(\alpha_r)>$ taken over $\binom{N}{n}$ possible samples is the corresponding symmetric mean of the population. Conversly, if a population parameter is written as a linear combination of symmetric means, then an unbiased estimator of a population symmetric mean is simply obtained by replacing each symmetric mean by the corresponding sample mean.

I. Two variate ratio type estimator when the population means of the auxiliary characters are known :

Let y_{ij} and x_{ij} be the observations on y and x respectively for the j-th second stage unit (s.s.u.) in the i-th primary sampling unit (p.s.u.) and x_{ij} be the corresponding value for x. Let N be the number of p.s.u's. in the population and M that of the s.s.u's, in each p.s.u. A simple random sample of n p.s.u's. is drawn first and from each of these n p.s.u's. in turn, a sample of m s.s.u's. is drawn by s.r.s. Using this sampling scheme the expression for unbiased two variate ratio estimator is derived following Tukey's notations. In the present study Two auxiliary characters viz., x_1 and x_2 have been considered. Thus there is a five-dimensional population given by,

$$(y_i, x_{1i}, x_{2i}, r_{1i}, r_{2i}),\ i=1, 2, \ldots N$$

where $$r_{1i}=\frac{y_i}{x_{1i}} \quad \text{and} \quad r_{2i}=\frac{y_i}{x_{2i}}.$$

Notations :

Let

$\bar{Y}_{NM}=<10000>''$ be the mean per element of y in the population

$\bar{X}_{1NM}=<01000>''$ be the mean per element of x_1 in the population

$\bar{r}_{1NM}=<00000>''$ be the mean of ratio of y to x_1 in the population

$\bar{y}_{nm}=\frac{1}{n}\sum_{i}^{n} <10000>_i$ be the mean per element of y in the sample

$\bar{x}_{1mn}=\frac{1}{n}\sum_{i}^{n} <01000>_i$ be the mean per element of x_1 in the sample

$\bar{r}_{1nm}=\frac{1}{n}\sum_{i}^{n} <00010>_i$ be the mean of ratio of y to x_1 in the sample

$\bar{Y}_i = <10000>'_i$ be the mean of y in i-th primary unit

$\bar{y}_{im} = <10000>_i$ be the sample mean of y for i-th p.s.u.

$\bar{x}_{1im} = <01000>_i$ be the sample mean of x_1 for i-th p.s.u.

$\bar{r}_{1im} = <00010>_i$ be the mean of the ratio of y to x_1 in the sample drawn from i-th p.s.u.

Similarly notations for auxiliary character x_2 can be defined.

The expression for covariance berween the estimators T_1 and T_2, based on x_1 and x_2 respectively Garg and Pillai [1] derived using the technique of symmetric means is given below,

$$\text{Cov}\,(T_1, T_2) = E\,(T_1, T_2) - \bar{Y}^2_{NM}$$

where $T_k = r_{knm}\,\bar{x}_{kNM}$

$$+\left\{1 + \frac{M-m}{NM\,(m-1)}\right\}\bar{y}_{nm} - \frac{n\,(N-1)}{N\,(n-1)}\,\bar{r}_{knm}\,\bar{x}_{knm}$$

$$-\frac{1}{nN}\left\{\frac{M-m}{M\,(m-1)} - \frac{N-n}{n-1}\right\}\sum_i^n \bar{r}_{kim}\,\bar{x}_{kim} \quad k=1, 2 \qquad \ldots(1)$$

Cov (T_1, T_2)

$$= E\,[\,\bar{r}_{1nm}\,\bar{x}_{1NM} + \left\{1 + \frac{M-m}{NM\,(m-1)}\right\}\bar{y}_{nm}$$

$$-\frac{n\,(N-1)}{N\,(n-1)}\,\bar{r}_{1nm}\,\bar{x}_{1nm}$$

$$\frac{1}{nN}\left\{\frac{M-m}{M\,(m-1)} - \frac{N-n}{n-1}\right\}\sum_i^n \bar{r}_{1im}\,\bar{x}_{1im}] \times [\bar{r}_{2nm}\,\bar{x}_{2NM}$$

$$+\left\{1 + \frac{M-m}{NM\,(m-1)}\right\}\bar{y}_{nm} - \frac{n\,(N-1)}{N\,(n-1)}\,\bar{r}_{2nm}\,\bar{x}_{2nm}$$

$$-\frac{1}{nN}\left\{\frac{M-m}{M\,(m-1)} - \frac{N-1}{n-1}\right\}\sum_i^n \bar{r}_{2im}\,\bar{x}_{2im}] - \bar{Y}^2_{NM}$$

The expression for Cov (T_1, T_2) for large N and M is given by

Cov (T_1, T_2)

$$=E(\bar r_{1nm}\,\bar x_{1NM}\,\bar r_{2nm}\,\bar x_{2NM})+E(\bar y_{nm}\,\bar r_{2nm}\,\bar x_{2NM})$$

$$-\frac{n}{n-1}E(\bar r_{1nm}\,\bar x_{1nm}\,\bar r_{2nm}\,\bar x_{2NM})$$

$$+\frac{1}{n(n-1)}E\left(\sum_i^n \bar r_{1im}\,\bar x_{1im}\,\bar r_{2nm}\,\bar x_{2NM}\right)$$

$$+E(\bar r_{1nm}\,\bar x_{1NM}\,\bar y_{nm})+E\left(\bar y_{nm}^2\right)-\frac{n}{n-1}E(\bar r_{1nm}\,\bar y_{nm}\,\bar x_{1nm})$$

$$+\frac{1}{n(n-1)}E\left(\bar y_{nm}\sum_i^n \bar r_{1im}\,\bar x_{1im}\right)$$

$$-\frac{n}{n-1}E(\bar r_{1nm}\,\bar x_{1NM}\,\bar r_{2nm}\,\bar x_{2nm})$$

$$-\frac{n}{n-1}E(\bar y_{nm}\,\bar r_{2nm}\,\bar x_{2nm})+\frac{n^2}{(n-1)^2}E(\bar r_{1nm}\,\bar x_{1nm}\,\bar r_{2nm}\,\bar x_{2nm})$$

$$-\frac{1}{(n-1)^2}E\left(\bar r_{2nm}\,\bar x_{2nm}\sum_i^n \bar r_{1im}\,\bar x_{1im}\right)$$

$$+\frac{1}{n(n-1)}E\left(\bar r_{1nm}\,\bar x_{1NM}\sum_i^n \bar r_{2im}\,\bar x_{2im}\right)$$

$$+\frac{1}{n(n-1)}E\left(\bar y_{nm}\sum_i^n \bar r_{2im}\,\bar x_{2im}\right)$$

$$-\frac{1}{(n-1)^2}E\left(\bar r_{1nm}\,\bar x_{1nm}\sum_i^n \bar r_{2im}\,\bar x_{2im}\right)$$

$$+\frac{1}{n^2(n-1)^2}E\left(\sum_i^n \bar x_{1im}\,\bar r_{1im}\sum_i^n \bar r_{2im}\,\bar x_{2im}\right)-\bar Y_{NM}^2 \quad \ldots(2)$$

These expectations have been solved with the help of symmetric means, as an illustration.

$$E(\bar y_{nm}\,\bar r_{2nm}\,\bar x_{2NM})=E\left(\frac{1}{n}\sum_i^n <10000>_i\,\frac{1}{n}\sum_i^n <00001>_i\right)$$

$$<00100>''$$

now applying the multiplication rule of symmetric means and simplifying we get.

$$E(\bar{y}_{nm}\,\bar{r}_{2nm}\,\bar{x}_{2NM})=\Big[\frac{1}{nm}\,\frac{1}{N}\sum_{i}^{N}<10001>'_i$$

$$+\frac{m-1}{nm}\,\frac{1}{N}\sum_{i}^{N}<(10000)\,(00001)>'_i$$

$$+\frac{n-1}{nN\,(N-1)}\Big\{\sum_{i}^{N}<10000>'_i\sum_{i}^{N}<00001>'_i$$

$$-\sum_{i}^{N}<10000>'_i<00001>'_i\Big\}\Big]<00100>''$$

Simplifying for large N and M we have,

$$E(\bar{y}_{nm}\,\bar{r}_{2nm}\,\bar{x}_{2NM})=\Big[\frac{1}{nm}<10001>''+\frac{m-1}{mn}<(10000)\,(00001>)''$$

$$+\frac{n-1}{n}<10000>''<00001>''\Big]<00100>''$$

Similarly other values of the expectations can be obtained. Substituting values of these expectations in equation (2), and reducing the symmetric means to the standard form we get,

$$\text{Cov}\,(T_1,T_2)=V\,(\bar{y}_{nm})+\bar{r}_{1NM}\,\bar{r}_{2NM}\,\text{Cov}\,(\bar{x}_{1nm},\bar{x}_{2nm})$$

$$-\bar{r}_{1NM}\,\text{Cov}\,(\bar{y}_{nm},\bar{x}_{1nm})-\bar{r}_{2NM}\,\text{Cov}\,(\bar{y}_{nm},\bar{x}_{2nm})$$

$$+\frac{n}{n-1}\,\text{Cov}\,(\bar{x}_{1nm},\bar{r}_{2nm})\,\text{Cov}\,(\bar{x}_{2nm},\bar{r}_{1nm})$$

$$+\frac{n}{n-1}\,\text{Cov}\,(\bar{x}_{1nm},\bar{x}_{2nm})\,\text{Cov}\,(\bar{r}_{1nm},\bar{r}_{2nm}\quad\ldots(3)$$

The general expression for the covariance between T_i and T_j estimators for k auxiliary variates can be obtained by replacing suffix 1 by i and 2 by j in equation (3) and is given by

$$\text{Cov}\,(T_i,T_j)=V_{ij}=V\,(\bar{y}_{nm})+\bar{r}_{iNM}\times\bar{r}_{jNM}\,\text{Cov}\,(x_{1nm},\bar{x}_{jnm})$$

$$-\bar{r}_{iNM}\,\text{Cov}\,(\bar{y}_{nm},\bar{x}_{inm})-\bar{r}_{jNM}\,\text{Cov}\,(\bar{y}_{nm},\bar{x}_{jnm})$$

$$+\frac{n}{n-1}\,[\text{Cov}\,(\bar{x}_{jnm},\bar{x}_{inm})\,\text{Cov}\,(\bar{r}_{inm},\,\bar{}_{jnm})$$

$$+\text{Cov}\,(\bar{x}_{inm}\,\bar{r}_{jnm})\,\text{Cov}\,(\bar{x}_{inm},\bar{r}_{inm})]$$

Unbiased ratio estimator for two auxiliary characters is as follows:

$$T_{M.R} = \sum_{k=1}^{2} w_k \, T_k$$

where w_k are the weights such that $\sum_{k=1}^{2} w_k = 1$ and T_k is given by eq. (1).

This unbiased two-variate ratio estimator can be put in the form of a matrix $T_{M.R} = W\,T_r$

where $$W = [\{w_1, w_2)]$$

and $$T_r = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

if V is defined as variance-covariance matrix then,

$$\begin{bmatrix} V(T_1) & \text{Cov}(T_1, T_2) \\ \text{Cov}(T_1, T_2) & V(T_2) \end{bmatrix} = V$$

where $$V(T_k) = V(\bar{y}_{nm}) + \bar{r}^2_{kNM}\, V(\bar{x}_{knm}) - 2\bar{r}_{kNM}\, \text{Cov}(\bar{y}_{nm}, \bar{x}_{knm}) + \frac{n}{n-1}\left[\{\text{Cov}(\bar{x}_{knm}, \bar{r}_{knm})\}^2 + \text{Var}(\bar{x}_{knm})\, V(\bar{r}_{knm})\right]$$

$$k = 1, 2.$$

and Cov (T_1, T_2) is given by eq. (3)

Now $$V^{-1} = \begin{bmatrix} V(T_2)/D & -\text{Cov}(T_1, T_2)/D \\ -\text{Cov}(T_1, T_2)/D & V(T_1)/D \end{bmatrix}$$

where $$D = V(T_1)\, V(T_2)) = \text{Cov}^2(T_1, T_2)$$

Then $$w_1 = \frac{V(T_2) - \text{Cov}(T_1, T_2)}{D_1}$$

and $$w_2 = \frac{V(T_1) - \text{Cov}(T_1, T_2)}{D_1}$$

and $V_{Min}(T_{MR}) = 1/D_1$

where D_1 is the sum of all the elements in the matrix V^{-1}.

II. Ratio-type estimators when the population means of auxiliary characters are not known.

Sometimes the population mean $\bar{x}_N$ is not known. In such cases $\bar{x}_N$ is first estimated from a large preliminary sample of n' p.s.u's. from each of which m' p.s.u's. are drawn. A sub sample of size n p.s.u's. is drawn from n' p.s.u's. and is each of these p.s.u's. m s;s.u's are drawn from m' s.s.u's. to observe y.

Let, $$\bar{x}_{1n'm'} = \frac{1}{n'} \sum_{i}^{n'} <01000>^*_i$$

be the mean per element for x_1 in the preliminary sample.

and $$\bar{x}_{1im'} = <01000>^*_i$$

be the mean per element for x_1 for the sample drawn from i-th p.s.u. in the preliminary sample. Similarly $\bar{x}_{2n'm'}$ and $\bar{x}_{2im'}$ can be defined.

Garg and Pillai [1] derived the expression for variance of the unbiased ratio estimate for one auxiliary character and information on the same is collected through double sampling. Here the case of two auxiliary characters has been considered and the expression for covariance between T_1' and T_2', the estimates based on x_1 and x_2 has been derived.

where $$T_k' = \bar{r}_{knm}\, \bar{x}_{kn'm'} + \left(1 + \frac{m'-m}{n'm'(m-1)}\right) \bar{y}_{nm} - \frac{n(n'-1)}{n'(n-1)} \bar{r}_{knm}\, \bar{x}_{knm}$$
$$- \frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right) \sum_{i}^{n} \bar{r}_{kim}\, \bar{x}_{kim} \quad k=1, 2 \quad \ldots(5)$$

and $$V(T_k') = V(\bar{y}_{nm}) + \bar{r}^2_{kNM} \operatorname{Var}(\bar{x}_{knm}) - \bar{r}^2_{kNM} \operatorname{Var}(\bar{x}_{kn'm'})$$
$$- 2\bar{r}_{kNM} \operatorname{Cov}(\bar{y}_{nm}, \bar{x}_{knm}) + 2\bar{r}_{kNM} \operatorname{Cov}(\bar{y}_{nm}, \bar{x}_{kn'm'}) \quad \ldots(6)$$

Now $$\operatorname{Cov}\left(T_1', T_2'\right) = E\left(T_1' \times T_2'\right) - \bar{Y}^2_{NM}$$
$$= E\Bigg[\bar{r}_{1nm}\, \bar{x}_{1m'n'} + \left(1 + \frac{m'-m}{n'm'(m-1)}\right) \bar{y}_{nm} - \frac{n(n'-1)}{n'(n-1)} \bar{r}_{1nm}\, \bar{x}_{1nm}$$
$$- \frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right) \sum_{i}^{n} \bar{r}_{1im}\, \bar{x}_{1im}\Bigg] \times \Bigg[\bar{r}_{2nm}\, \bar{x}_{2n'm'}$$
$$+ \left(1 + \frac{m'-m}{n'm'(m-1)}\right) \bar{y}_{nm} - \frac{n(n'-1)}{n'(n-1)} \bar{r}_{2nm}\, \bar{x}_{2nm}$$
$$- \frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right) \sum_{i}^{n} \bar{r}_{2im}\, \bar{x}_{2im}\Bigg]$$

$$\text{Cov}\left(T_1', T_2'\right)=E\ (\bar{r}_{1nm}\ \bar{r}_{2nm}\ \bar{x}_{1n'm'}\ \bar{x}_{2n'm'})$$

$$+\left(1+\frac{m'-m}{n'm'(m-1)}\right) E\ (\bar{y}_{nm}\ \bar{r}_{2nm}\ \bar{x}_{2n'm'})$$

$$-\frac{n(n'-1)}{n'(n-1)} E\ (\bar{r}_{1nm}\ \bar{x}_{1nm}\bar{r}_{2n'm'}\ \bar{x}_{2n'm'})-\frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)}-\frac{n'-n}{n-1}\right)$$

$$\times E\left(\bar{r}_{2nm}\ \bar{x}_{2n'm'}\sum_i^n \bar{r}_{1im}\ \bar{x}_{1im}\right)-\frac{n(n'-1)}{n'(n-1)} E\ (\bar{r}_{1nm}\ \bar{x}_{1n'm'}\ \bar{r}_{2nm}\ \bar{x}_{2nm}$$

$$-\frac{n(n'-1)}{n'(n-1)}\left(1+\frac{m'-m}{n'm'(m-1)}\right) E\ (\bar{y}_{nm}\ \bar{r}_{2nm}\ \bar{x}_{2nm})+\left(\frac{n(n'-1)}{n'(n-1)}\right)^2$$

$$\times\ E\ (\bar{r}_{1nm}\ \bar{x}_{1nm}\ \bar{r}_{2nm}\ \bar{x}_{2nm})+\frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)}-\frac{n'-n}{n-1}\right)-$$

$$\frac{n(n'-1)}{n'(n-1)}\ \times E\left(\bar{r}_{2nm}\ \bar{x}_{2nm}\sum_i^n \bar{r}_{1im}\ \bar{x}_{1im}\right)+\left(1+\frac{m'-m}{n'm'(m-1)}\right)$$

$$E\ (\bar{r}_{1nm}\ \bar{x}_{1n'm'}\ \bar{y}_{nm})$$

$$+\left(1+\frac{m'-m}{n'm'(m-1)}\right)^2 E\left(\bar{y}^2_{nm}\right)-\frac{n(n'-1)}{n'(n-1)}\left(1+\frac{m'-m}{n'm'(m-1)}\right)$$

$$E(\bar{r}_{1nm}\ \bar{x}_{1nm}\ \bar{y}_{nm})$$

$$-\frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)}-\frac{n'-n}{n-1}\right)\left(1+\frac{m'-m}{n'm'(m-1)}\right)$$

$$E\left(\sum_i^n \bar{r}_{1im}\ \bar{x}_{1im}\ \bar{y}_{nm}\right)$$

$$-\frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)}-\frac{n'-n}{n-1}\right) E\left(\bar{r}_{1nm}\ \bar{x}_{1n'm'}\sum_i^n \bar{r}_{2im}\ \bar{x}_{2im}\right)$$

$$-\frac{1}{nn'}\left(\frac{m'-m)}{m'(m-1)}-\frac{n'-n}{n-1}\right)\left(1+\frac{m'-m}{n'm'(m-1)}\right)$$

$$E\left(\bar{y}_{nm}\ \sum_i^n \bar{r}_{2im}\ \bar{x}_{2im}\right)$$

$$+\frac{1}{nn'}\left(\frac{m'-m}{m'(m-n)}-\frac{n'-n}{n-1}\right)\frac{n(n'-1)}{n'(n-1)}$$

$$E\left(\bar{r}_{1nm}\ \bar{x}_{1nm}\ \sum_i^n \bar{r}_{2im}\ \bar{x}_{2im}\right)$$

$$+\left\{\frac{1}{nn'}\left(\frac{m'-m}{m'(m-1)}-\frac{n'-n}{n-1}\right)\right\}^2 .$$

$$E\left(\sum_i^n \bar{r}_{1im}\,\bar{x}_{1im}\sum_i^n \bar{r}_{2im}\,\bar{x}_{2im}\right)-\bar{Y}^2_{NM}$$

Simplifying these expectations with the help of symmetric means for large N and M and retaining the terms of order $1/nm$ and and $1/n'm'$ only and reducing the symmetric means to standard form we get,

$$\text{Cov}\left(T_1',\ T_2'\right)$$

$$=V(\bar{y}_{nm})+\bar{r}_{1NM}\,\bar{r}_{2NM}\,\text{Cov}\,(\bar{x}_{1nm},\,\bar{x}_{2nm})$$
$$-\bar{r}_{1NM}\,\bar{r}_{2NM}\,\text{Cov}\,(\bar{x}_{1n'm'},\,\bar{x}_{2n'm'})-\bar{r}_{1NM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{1nm})$$
$$-\bar{r}_{2NM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{2nm})+\bar{r}_{1NM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{1n'm'})$$
$$-\bar{r}_{NM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{2n'm'}) \quad ...(7)$$

The general expression for covariance based on p auxiliary characters in double sampling can be written by replacing the suffix 1 by i and 2 by j in eq. (7) and is given by,

$$\text{Cov}\left(T_i',\ T_j'\right)$$

$$=V(\bar{y}_{nm})+\bar{r}_{1NM}\times\bar{r}_{jNM}\,\text{Cov}\,(\bar{x}_{inm};\,\bar{x}_{jnm})$$
$$-\bar{r}_{iNM}\,\bar{r}_{jNM}\,\text{Cov}\,(\bar{x}_{in'm'},\,\bar{x}_{jn'm'})-\bar{r}_{iNM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{inm})$$
$$-\bar{r}_{jNM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{jnm})+\bar{r}_{iNM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{in'm'})$$
$$-\bar{r}_{jNM}\,\text{Cov}\,(\bar{y}_{nm},\,\bar{x}_{jn'm'})$$

As in section 1 the multivariate unbiased ratio estimator based on two auxiliary characters in double sampling has also been considered *viz.*,

$$T'_{MRD}=\sum_{k=1}^{n} w_k'\quad T_k'$$

where w_k' are the weights such that $\sum_{k=1}^{2} w_k'=1.$

Here, $w_k'=R_i/D'$ where R_i is the total of all the elements in the i-th row of the inverse of A, the variance-covariance matrix of the estimates T'_k ($k=1, 2$) given by eq. (5) and D' $(=\Sigma R_i)$ is the sum of all elements of A^{-1} and the minimum variance $V\left(T'_{MRD}\right)=1/D'$ This can be generalised for k auxiliary variates.

Results and Discusison

The results obtained above have been illustrated with the help of data from 1961 population census of PanchMohal, Baroda, Broach and Surat district in Gujarat State. Taluks were taken as psu's and villages within taluks as ssu's. The village wise number of agricultural workers was taken as y and the corresponding number of households and total population as x_1 and x_2 respectively.

The relative efficiency of the various biased and unbiased ratio estimators based on x_1 and x_2 compared to $\bar{y}_{nm}$ when their population means are known is given in table 1.

TABLE 1

Estimator	*Relative efficiency*
1. Simple estimate $(\bar{y}_{nm})$	100.0
2. (*a*) Ratio estimate based on x_1 $(\hat{y}_{R_1})$	123.6
3. (*b*) Ratio estimate based on x_2 $(\hat{y}_{R_2})$	127.5
4. (*a*) Unbiased ratio estimate (T_1)	124.0
(*b*) Unbiased ratio estimate (T_2)	127.7
(*c*) Unbiased two variate ratio estimate (T_{MR})	133.3

It may be noted that the relative efficiency of T_{MR} is higher than that of ratio/unbiased ratio estimators based on either of the auxiliary character.

In table 2 the efficiencies of the estimators T'_1 and T'_2 relative to those of $\hat{y}_{R_1}$ and $\hat{y}_{R_2}$ respectively and of T'_{MRD} relative to those of

$\hat{y}_{R1}$ or $\hat{y}_{R2}$ have been given for the case when population means of x_1 and x_2 are not known.

TABLE 2

Relative efficiency of various estimators

Type of estimators	*Relative efficiency*
$\dfrac{V(\hat{y}_{R1})}{V\left(T'_1\right)} \times 100$	127.2
$\dfrac{V(\hat{y}_{R2})}{V\left(T'_2\right)} \times 100$	122.4
$\dfrac{V(\hat{y}_{R1})}{V\left(T'_{MRD}\right)} \times 100$	133.0
$\dfrac{V(\hat{y}_{R2})}{V\left(T'_{MRD}\right)} \times 100$	127.3

It is observed that the relative efficiency of T'_1 and T'_2 is higher as compared to corresponding biased ratio seatimators. It may be further noted that the relative efficiency of T'_{MRD} is higher than that of ratio/unbiased ratio estimators based on either auxiliary characters.

SUMMARY

Using the expression for covariance between unbiased ratio estimators using x_1 and x_2 (auxiliary characters) derived in the case of two stage designs, expressions for variances of unbiased two-auxiliary variate ratio estimators of $\bar{y}_N$ have been derived in a two stage design when (*i*) the population means of x_1 and x_2 are known, and (*ii*) population means of x_1 and x_2 are not known (double sampling), using the technique of symmetric means for large values

of N and M. The results obtained are illustrated with the help of 1961 population census data on number of agricultural workers for four districts of Gujarat State (India). It was found that the unbiased two auxiliary variate ratio estimator was the most efficient among the estimators considered in both the cases.

ACKNOWLEDGEMENT

The authors are thankful to the refree for the suggestions in improving the paper.

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